https://www.linkedin.com/feed/update/urn:li:activity:6639742068216463360

If $a_1 = 1$ and $a_n = n(1 + a_{n-1}), \forall n \geq 2,$ then

$$\lim_{n\to\infty}\left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right)\ldots\left(1 + \frac{1}{a_n}\right) = ?$$

Solution by Arkady Alt, **San Jose** ,**California**, **USA**.

Since $a_{n+1} = (n+1)(1+a_n) \Leftrightarrow \dfrac{a_{n+1}}{a_n(n+1)} = \left(1 + \dfrac{1}{a_n}\right), \forall n \in \mathbb{N}$ then

$$\prod_{k=1}^{n}\left(1 + \frac{1}{a_k}\right) = \prod_{k=1}^{n}\frac{a_{k+1}}{a_k(k+1)} = \frac{a_{n+1}}{a_1(n+1)!} = \frac{a_{n+1}}{(n+1)!}.$$

From the other hand, since $a_{n+1} = (n+1)(1+a_n) \Leftrightarrow a_{n+1} = (n+1)a_n + n + 1 \Leftrightarrow$

$\dfrac{a_{n+1}}{(n+1)!} = \dfrac{a_n}{n!} + \dfrac{1}{n!}, \forall n \in \mathbb{N}$, we obtain

$\dfrac{a_{n+1}}{(n+1)!} - \dfrac{a_1}{1!} = \displaystyle\sum_{k=1}^{n}\left(\frac{a_{k+1}}{(k+1)!} - \frac{a_k}{k!}\right) = \sum_{k=1}^{n}\frac{1}{k!} \Leftrightarrow \frac{a_{n+1}}{(n+1)!} = \sum_{k=0}^{n}\frac{1}{k!}$ and, therefore,

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1 + \frac{1}{a_k}\right) = \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)!} = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} = e.$$